Nonlinear Robust Optimal Control via Inverse Problem of Optimal Regulator

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Abstract

Recently, an approach to derive solutions of Hamilton-Jacobi equation with high accuracy, called *stable manifold theory*, is proposed. So far some researches about nonlinear optimal control via stable manifold are subjected. However, research of robustness for nonlinear optimal control is not reported. This report proposes a new framework to enhance robustness to the nonlinear optimal control system designed by the stable manifold method using the inverse problem of optimal regulator and linear matrix inequality. To verify the effectiveness, the swing-up and stabilization problem for the Acrobot is considered. The Acrobot is underactuated mechanical system, which is composed of two links. The system is typical problem in nonlinear control theory. Simulation results are presented in order to evaluate the control performance.

1 Introduction

The underactuated mechanical systems is the system which has actuators less than drive parts. It is effective for lowering the cost, weight reduction and saving energy to achieve the control objective by a few actuators. However, it is very difficult for underactuated mechanical systems with nonlinear characteristics to control. Such as auto mobile, aircraft, helicopter, watercraft and spacecraft are known as underactuated mechanical systems. The Acrobot, which is discussed in this thesis, is one of the representative example and considered as typical problem for evaluation of control performance in nonlinear control theory. Swing up and stabilized control of the Acrobot is attained by hybrid control of partial feedback linearization and linear control theory[1]. There are other approaches to conduct swing up control such as reinforcement learning, which is the method by which manipulator obtain the behavior by itself [2], energy feedback [3], backstepping method [4]. Furthermore, attitude control of the Acrobot by sum of square method is discussed [5]. In this paper, we propose an approach about swing up control with framework of nonlinear optimal control theory via *stable manifold theory* [6]. *Stable manifold theory* is method to obtain the solution of Hamilton-Jacobi equation in high accuracy. Nonlinear optimal control theory using *stable manifold theory* is applied to aircraft stall recovery [7], pilot induced oscillation restraint [8] and magnetic levitation system [9]. Also, the approach is applied to swing up control of the Acrobot [10].

Actuality systems have uncertainties such as parameter fluctuating by aging, disturbance, system noise and depending on the environment. There are some report about nonlinear optimal control theory even though the theory, which guarantee robustness against uncertainties, is not reported. To guarantee robustness against uncertainties is very important problem in terms of safety. Therefore, robust control theory recently has been researched. A method via linear matrix inequalities(LMI) is one of robust control theory [11].

In this report, nonlinear optimal control via *stable manifold theory*, LMI, inverse problem of optimal control are applied to nonlinear controller in order to guarantee robustness against uncertainties of parameters.

2 Inverse Problem of Optimal Control

In this section, inverse problem is reviewed. Linear system and evaluated function of quadratic form are considered as Eq.(1).

$$\begin{aligned} \Sigma \ : \ \dot{x} &= Ax + Bu \\ J &= \int_0^\infty (x^T Q x + u^T R u) dt \end{aligned} \tag{1}$$

Where $x \in \mathbb{R}^n$ represents state variables, $u \in \mathbb{R}^m$ represents control inputs, $Q = Q^T \succeq 0$, $R \succ 0$ shows weight matrix. Inverse problem of optimal regulator is the method which derive weight matrix of evaluated function J that is minimized by feedback gain K when control input $u = -Kx$ of state feedback is given for linear system Σ.

2.1 Inverse problem of optimal regulator [12]

Let control input $u = -Kx$ is applied to Eq.(1). Then weight matrix of minimized evaluated function is obtained. First, we assume following contents.

Assumption 1

1. Linear system Σ is controllable.
2. Closed loop system $\dot{x} = (A - BK)x$ is stable.
3. A weight matrix R is identity matrix.

Following contents are given when (A, B) is possible to stabilize.

(A) Feedback gain K is optimal and stable.

(B) (C, A) is observable also $P \succeq 0$ and C that satisfy $PA + A^T P - PBR^{-1}B^T P + C^T C = 0$ and $K = B^T P$ exist

Following matrix is introduced.

$$\begin{aligned} \Gamma &\equiv \begin{bmatrix} PA + A^T P - K^T K & PB - K^T \\ B^T P - K & 0 \end{bmatrix} \\ &= -\begin{bmatrix} C^T \\ 0 \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix}, \ (P = P^T) \end{aligned} \tag{2}$$

Eq.(2) is obtained by transformed Riccati equation and $K = B^T P$. Therefore, symmetrical solution of Riccati equation P that satisfy $PA + A^T P - PBR^{-1}B^T P + C^T C = 0$ and $K = B^T P$ satisfy following condition.

$$\Gamma(P) \preceq 0 \tag{3}$$

Weight matrix Q is derived from the above result as follows.

1) To derive symmetric solution P which satisfy Eq.(3).

2) Calculating following matrix with solution P in procedure 1) to derive Q.

$$Q = K^T K - PA - A^T P \tag{4}$$

3) To confirm (C, A) is observable when Q is partitioned as $Q = C^TC$. (However procedure 3) is unnecessary when feedback gain K satisfy $\lambda(A) - \lambda(A - BK) \neq 0$. $\lambda(\cdot)$ represents eigenvalues of matrix.)

3 Nonlinear Optimal Control Theory

In this section, a nonlinear partial differential equation called Hamilton-Jacobi equation which is used nonlinear optimal control problem is derived. Let nonlinear system Σ_{nl} is considered.

$$\Sigma_{\text{nl}} \ : \ \dot{x} = f(x) + g(x)u \tag{5}$$

Where $x \in \mathbb{R}^n$ represents state variables, $u \in \mathbb{R}^m$ represents control inputs. $f(x) = 0$ is formed at equivalent point $(x = 0)$. Then evaluated function Eq.(1) is considered for Eq.(5). Following Hamiltonian is obtained by using dynamic programming.

$$\begin{aligned} H(x, \frac{\partial V(x)}{\partial x}, u) &= \frac{\partial V}{\partial x}\dot{x} + x^TQx + u^TRu \\ &= \frac{\partial V(x)}{\partial x}(f(x) + g(x)) + x^TQx + u^TRu \end{aligned} \tag{6}$$

Then $V(x)$ is function which is $V(x) > 0$, $V(0) = 0$. A optimal input u^* which minimize evaluated function is derived by partial differentiating Hamiltonian by u because Hamiltonian is downward convex function for u.

$$u^* = -\frac{1}{2}R^{-1}g(x)^T\frac{\partial V(x)}{\partial x}^T \tag{7}$$

Hamilton-Jacobi equation is obtained by substituting a optimal input u^* for Eq.(6).

4 The Acrobot System

The Acrobot consists of first link in free motion, second link in active motion by actuator. In this section, mathematical model of the Acrobot is described.

4.1 Mathmatical Model

A model of the Acrobot is shown as Fig.1. An actuator is mounted between first link and second link. Let q_i,m_i,J_i,L_i,L_{Ci}, $(i = 1, 2)$ represent angle, weight,

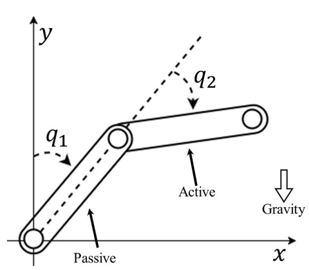

Figure 1 Model of the Acrobot

moment of inertia, length, distance between center of gravity and axis of rotation of first link and second link respectively. g represents gravitational acceleration. Kinematic energy $K(q, \dot{q})$ and potential energy $U(q)$ are as follows.

$$\begin{aligned} K(q, \dot{q}) &= \frac{1}{2}\dot{q}^TM(q_2)\dot{q} \\ U(q) &= b_1\cos(q_1) + b_2\cos(q_1 + q_2) \ , \ \left(q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) \end{aligned} \tag{8}$$

Then $M(q_2)$ given as Eq.(9).

$$M(q_2) = \begin{bmatrix} a_1 + a_2 + 2a_3\cos(q_2) & a_2 + a_3\cos(q_2) \\ a_2 + a_3\cos(q_2) & a_2 \end{bmatrix} \tag{9}$$

$$\begin{aligned} &a_1 = m_1L_{C1}^2 + m_2L_1^2 + J_1, \ a_2 = m_2L_{C2}^2 + J_2 \\ &a_3 = m_2L_1L_{C2}, \ b_1 = (m_1L_{C1} + m_2L_1)g \\ &b_2 = m_2L_{C2}g \end{aligned}$$

Kinematic equation is derived as Eq.(10) due to kinematic equation of Euler-Lagrange $d(\partial L/\partial\dot{q}) = (\partial L/\partial q)$ when Lagrangean $L(q, \dot{q}) = K(q, \dot{q}) - U(q)$.

$$M(q_2)\ddot{q} + N(q, \dot{q})\dot{q} + C(q) = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \tag{10}$$

$$N(q, \dot{q}) = \begin{bmatrix} -a_3\dot{q}_2\sin(q_2) & -a_3(\dot{q}_1 + \dot{q}_2)\sin(q_2) \\ a_3\dot{q}_1\sin(q_2) & 0 \end{bmatrix}$$

$$C(q) = \begin{bmatrix} -b_1\sin(q_1) - b_2\sin(q_1 + q_2) \\ -b_2\sin(q_1 + q_2) \end{bmatrix}$$

Motor torque τ for second link is described as Eq.(11) by taking counter electromotive force and viscous friction into consideration.

$$\tau = nK_{DC}u - \mu_2\dot{q}_2 \tag{11}$$

Parameters of the Acrobot refer to document [10]. Each parameters shown as Table 1.

Table 1 System parameters of the Acrobot

m_1[kg]	0.851	m_2[kg]	0.420
L_1[m]	0.162	L_2[m]	0.210
L_{C1}[m]	-0.017	L_{C2}[m]	0.076
J_1[kg ·m^2]	7.02×10^{-3}	J_2[kg ·m^2]	4.24×10^{-3}
n[/]	48/14	K_{DC}[N·m/V]	0.0196
μ_2[N ·m·s]	0.015	g[m/s^2]	9.81

4.2 State Equation

In this section, state equation is derived from Eq.(10). Nonlinear state equation is obtained as Eq.(12) due to state variable $x = [q_1, \ q_2, \ q_3, \ q_4]^T = [x_1, \ x_2, \ x_3, \ x_4]^T$.

$$\dot{x} = f(x) + g(x)u \tag{12}$$

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ -M(q_2)^{-1}(N(q, \dot{q})\dot{q} + C(q) \begin{bmatrix} 0 \\ \mu_2x_4 \end{bmatrix} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ M(q_2)^{-1}\begin{bmatrix} 0 \\ nK_{DC} \end{bmatrix} \end{bmatrix}$$

Furthermore, linear state equation derived by linearizing Eq.(12) origin as Eq.(13).

$$\dot{x} = Ax + Bu \tag{13}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 19.8539 & -16.4912 & 0 & 2.0820 \\ 11.1251 & 73.1614 & 0 & -5.9071 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ -7.1382 \\ 20.2528 \end{bmatrix}$$

5 Controller Design

In this research, nonlinear controller which has robustness for uncertainty of parameters is designed. The approach of proposed method is shown as follows.

(i) To design robust LQ controller by using Eq.(13).

(ii) Deriving weight matrix by solving inverse problem.

(iii) To solve nonlinear optimal control problem with weight matrix in procedure (ii).

5.1 Robust LQ Control

Regulator problem is considered, which is minimized following evaluated function. Weight matrix is determined by trial and error.

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \tag{14}$$

$$Q = \begin{bmatrix} 0.005 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}, \ R = 1$$

In this research, robust controller is designed on the assumption that uncertainty of friction coefficient μ_2. μ_2 is defined as follows.

$$\forall \mu_2 \in \begin{bmatrix} \underline{\mu_2} & \overline{\mu_2} \end{bmatrix}, \ (:= [0.0147 \quad 0.0153]) \tag{15}$$

Then matrix A is given as Eq.(16).

$$\forall A(\mu_2) \in [A_1 \quad A_2], \ (:= \begin{bmatrix} A(\underline{\mu_2}) & A(\overline{\mu_2}) \end{bmatrix}) \tag{16}$$

Where Eq.(16) is arbitrary matrix, so infinite LMI conditions is required but it is generally known that they can be replaced into finite LMI conditions as Eq.(17) [13]. A state feedback gain is defined as $K_r = \mathrm{F}\mathrm{X}^{-1}$ by minimizing γ in the range where $\mathrm{X} = \mathrm{X}^T \succ 0$ and F which satisfies LMI condition Eq.(17)

$$\begin{aligned} &\text{minimize} : \gamma \\ &\text{subject to} : \\ &\begin{bmatrix} \mathrm{He}[A_i \mathrm{X} + B\mathrm{F}] & \mathrm{X}Q_h^T & \mathrm{F}^T R \\ Q_h \mathrm{X} & I_{4\times4} & 0_{4\times1} \\ R\mathrm{F} & 0_{1\times4} & R \end{bmatrix} \succ 0 \\ &(i = 1, 2) \\ &\begin{bmatrix} \mathrm{Z} & I_{4\times4} \\ I_{4\times4} & \mathrm{X} \end{bmatrix} \succ 0, \ \mathrm{trace}[\mathrm{Z}] \preceq \gamma \end{aligned} \tag{17}$$

A feedback gain which is given by solving Eq.(17) is described as Eq.(18).

$$K_r = [460.8546 \quad 180.1654 \quad 94.4013 \quad 34.3566] \tag{18}$$

5.2 Inverse Problem

In this section, weight matrix is derived with Eq.(18). First, following matrix inequality is introduced.

$$\begin{aligned} \Gamma(P_r) &= \begin{bmatrix} P_r A + A^T P_r - K_r^T K_r^T & P_r B - K_r \\ B^T P_r - K_r & 0 \end{bmatrix} \\ &= -\begin{bmatrix} C_r^T \\ 0 \end{bmatrix} [C_r \quad 0] \preceq 0 \end{aligned} \tag{19}$$

C_r is a matrix which satisfies $Q_r = C_r^T C_r$. Symmetric solution P_r is derived by solving Eq.(19)

$$P_r = 10^4 \times \begin{bmatrix} 2.1711 & 0.8167 & 0.4456 & 0.1593 \\ 0.8167 & 0.3076 & 0.1676 & 0.0600 \\ 0.4456 & 0.1676 & 0.0914 & 0.0327 \\ 0.1593 & 0.0600 & 0.0327 & 0.0117 \end{bmatrix} \tag{20}$$

Furthermore, a weight matrix is derived as follows by Eq.(21) with result of Eq.(20).

$$Q_r = -P_r A - A^T P_r + P_r B B^T P_r \tag{21}$$

$$Q_r = \begin{bmatrix} 16.8159 & 7.3526 & 0.3830 & 0.0428 \\ 7.3526 & 3.2171 & 0.1676 & 0.0186 \\ 0.3830 & 0.1676 & 0.3974 & 0.0568 \\ 0.0428 & 0.0186 & 0.0568 & 0.0086 \end{bmatrix}$$

5.3 Nonlinear Robust Optimal Control

Nonlinear optimal control problem is solved by using weight matrix Q_r which is given section 5.2. Hamiltonian $H(x, \partial V/\partial x, u)$ is obtained as Eq.(22).

$$\begin{aligned} H(x, \frac{\partial V(x)}{\partial x}, u) =& \frac{\partial V(x)}{\partial x}(f(x) + g(x)u) \\ &+ x^T Q_r x + u^T R u, \ (R = 1) \end{aligned} \tag{22}$$

Then optimal control input is given as Eq.(23), which minimizes evaluated function.

$$u^* = -\frac{1}{2} g(x)^T \frac{\partial V(x)}{\partial x}^T \tag{23}$$

Where p is defined as $(\partial V/\partial x)^T$ for function V. Hamilton-Jacobi equation is given as Eq.(24).

$$\begin{aligned} H^*(x, p) =& f(x)^T p - \frac{1}{4} p^T g(x) g(x)^T p + x^T Q_r x \\ =& 0 \end{aligned} \tag{24}$$

Hamilton's canonical equation is given as Eq.(25) for Eq.(24).

$$\dot{x} = \frac{\partial H^*(x, p)}{\partial p}, \ \dot{p} = -\frac{\partial H^*(x, p)}{\partial x} \tag{25}$$

A solution p of Hamilton's canonical equation equivalent to partial difference $\partial V/\partial x$ of a solution of Hamilton-Jacobi equation, which is known in [6]. Therefore, optimal control input as Eq.(7) is replaced as follows.

$$u^* = -\frac{1}{2} g(x^*)^T p^* \tag{26}$$

Where x^*, p^* is represented solutions of Hamilton's canonical equation. As the result of above swing up trajectory is shown as Fig.2. A red line represents nominal swing up trajectory. It is possible that the Acrobot system is destabilization when the system is controlled by a red line trajectory only. Therefore blue line trajectories are calculated for robustness which is concerned in error of trajectory. Accuracy of calculation is verified by Hamiltonian value. Only if trajectory is solution of Hamilton-Jacobi equation, Hamiltonian value becomes sufficiently small. Hamiltonian value of Fig.2 is 6.1×10^{-6} at the most.

6 Simulation Result

The optimal input u^* is approximated by a polynomial to simulate of swing up control of the Acrobot. An initial condition is determined as $x(0) = [\pi, \ 0, \ 0, \ 0]^T$ for simulation. The initial condition represents situation which is first link and second link that is hang down together. Time response of state variable and input are shown in Fig.3, 4 respectively. Achievement of swing up control can be confirmed from the simulation result.

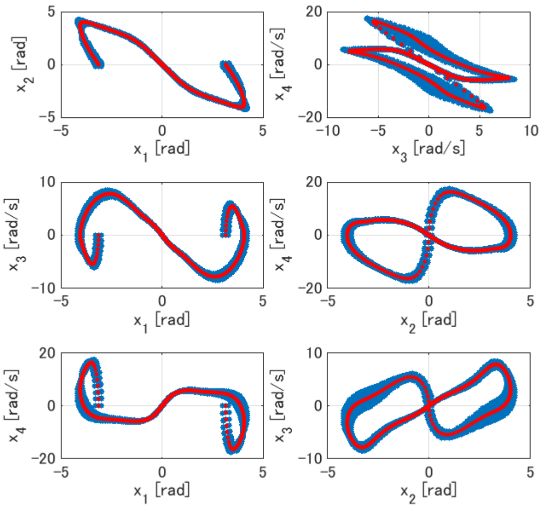

Figure 2 Trajectory of swing up motion.

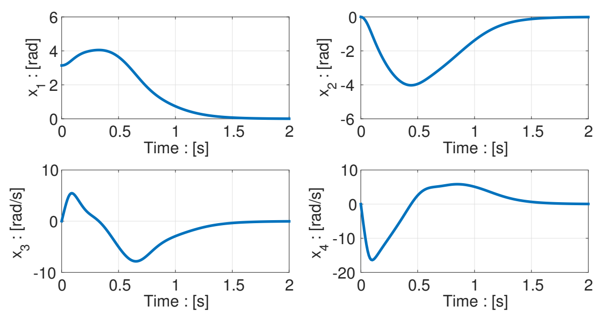

Figure 3 Time response of state variable.

7 Conclusion

In this research, we proposed a new approach of nonlinear control which is guaranteed robustness for uncertainty of parameters. It is based on characteristic that Hamilton-Jacobi equation is equivalent to Riccati equation in region of linear only. Furthermore, a proposed method is applied for swing up control of the Acrobot. Performance of the controller is verified on simulation. We make an experiment for verification of control performance as a future objective.

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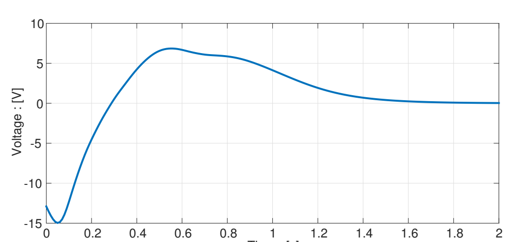

Figure 4 Time response of control input.

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