Partial Gain Scheduled Controller via Two Stage Design

M2015SC014 Akio SAMBAYASHI
Supervisor : Isao TAKAMI

Abstract

This paper proposes a new concept gain scheduled (GS) controller and its design method. For designing this controller, we proposed Two-stage design method and Pseudo ILQ. The effectiveness of proposed method is illustrated by simulations.

1 Introduction

It is natural for plants in the real world that dynamic characteristics change due to parameter fluctuations. Gain Scheduled (GS) controller gives the optimum control gain according to the fluctuation of parameters, and high controllability can be expected [1]. This method can be applied to various controlled objects [2]. However, GS requires a large amount of computation, which is a hindrance real-time control system design. Therefore, only the gain having the effect of gain scheduling is set as a variable gain based on the scheduling parameter. We propose a GS control system with fixed gain for a part of state variables. This is called Partial Gain Scheduled (PGS) controller. This reduces the amount of calculation and realizes controllability equivalent to that of the conventional GS. This paper shows the conditions of the plant to which this control system can be applied, but it is applicable to many plants. Next, the control system design method is shown. We adopt a two stage design method. This design method is derived as a theorem using Lienar Matrix Inequalities (LMI). At this time, in order to guarantee control performance, Inverse Optimum Control of Linear Quadratic Regulator design theory (ILQ) is applied [4]. Finally, we apply the proposed method to the jib crane system and show its effectiveness.

2 The formulation of the target system

There are restrictions on the target plant which can apply the proposed controller. A form of the target plant is given in this section. Consider a following system described by:

$$\Phi : \dot{x}(t) = A(\delta)x(t) + Bu(t) \tag{1}$$

$$A(\delta) = \begin{bmatrix} A_r(\delta) & A_{sr}(\delta) \\ 0 & A_s(\delta) \end{bmatrix} \quad B = \begin{bmatrix} B_r \\ B_s \end{bmatrix}$$

$$A_j(\delta) = A_{j0} + \sum_{i=1}^{k} \delta_i A_{ji} \quad (j \in \{r,\, sr,\, s\}),$$

where $A(\delta) \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. This system has a subsystem defined as Eq. (2):

$$\Phi_s : \dot{x}_s(t) = A_s(\delta)x_s(t) + B_s u_s(t), \tag{2}$$

where $A_s(\delta) \in \mathbb{R}^{n_1\times n_1}$, $B_s \in \mathbb{R}^{n_1\times m_1}$, ($n_1 < n$, $m_1 < m$). Then x is defined as $x := [x_r\, x_s]$. ($A(\delta)$, B) and ($A_s(\delta)$, B_s) are controllable. Eq. (1) has affine perturbation in each coefficient matrix, where $\delta_i \in \mathbb{R}$ is perturbation element which satisfy $|\delta_i| \le 1$. Let the target system and the subsystem be Φ and Φ_s, respectively. The target system and the subsystem which don't include δ by uniquely determining the value of δ are described as $\Phi(N)$ and $\Phi_s(N)$, respectively.

Remark 1 *When the state variable is selected, such as to satisfy the following assumptions, the proposed controller can be designed .*

- *The linear system has $A(\delta)$ such that the upper right or the lower left of the block matrices are zero.*
- *The subsystem is controllable.*

3 Two stage design for Partial GS controller

The obtained system in previous section has some time-varying parameters which affects largely the target plant behavior. As a first stage for proposed controller, in order to deal with this problem, a part of the target plant which includes some or all the scheduling parameters is defined as a subsystem. A closed-loop system is designed by using GS controller. As a second stage, a fixed controller which guarantees the robust stability and control performance is designed for target plant with the closed-loop subsystem. Through the above two stages, varying gains (which are defined as K_s) for the subsystem and fixed gains (which are defined as K_{cl}) for the target system which include the closed-loop for subsystem. The proposed controller is obtained by the sum of $K_s(\delta)$ and K_{cl}. As a result, the proposed controller has varying gains for a part of the state variable. In the first stage and the second stage, the controller is designed such that the following cost functions, Eq.(3) and Eq.(4), are minimized, respectively.

$$J = \int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \tag{3}$$

$$J_s = \int_0^\infty \left(x_s^T(t)Q_s x_s(t) + u_s^T(t)R_s u_s(t)\right) dt \tag{4}$$

These two cost functions need to be consistent in the whole system. In the first-stage design, to derive the stabilizing feed back gain scheduled controller $u_s = K_s(\delta)x_s(t)$ for the subsystem, the following conditions are well known to minimize the cost function Eq. (4).

$$P_s(A_s(\delta) + B_sK_s(\delta)) + (A_s(\delta) + B_sK_s(\delta))^T P_s + K_s^T(\delta)RK_s(\delta) < 0 \tag{5}$$

$$J_s = trace\,[P_s] < \gamma_s \tag{6}$$

If there exists $P_s = (P_s)^T > 0$ satisfying above matrix inequality, the subsystem, Eq. (2), is stabilized by $u_s = K_s(\delta)x_s$ and $J < \gamma$. Now, Eq. (5) is formulated as LMI by using congruence transformation, variable transformation and Schur compliment [1]. Let X_s and Y_s be $X_s := P_s^{-1}$ and $Y_s := K_sX_s$. Then the following matrix inequality is obtained by post-multiplying X_s^T and using Schur complement.

$$trace[X_s^{-1}] < \gamma_s \tag{7}$$

For (7), Z_s is introduced as an upper bound for X_s^{-1} since it is difficult to solve convex optimization problem. Then (7) is trance formed to Eq. (8) by using Schur complement.

$$\left[\begin{array}{cc} Z_s & I \\ I & X_s \end{array}\right] > 0,\ trace[Z_s] < \gamma_s \tag{8}$$

By using above conditions, the obtained class of gain scheduled controller is a linear function of scheduling parameters. By defining both Lyapnov matrix X_s and that of variable matrix Y_s as linear functions of scheduling parameters, the obtained class of that is easily expanded to a polynomial rational function. These design methods of GS controller will be described later. In the second stage, the proposed controller which is described by the sum of K_s and K_{cl} such that the above the cost function of Eq. (4) is minimized is designed. With the closed-loop subsystem by using varying gains of $K_{se}(\delta)$, the target plant is obtained as the following plant for the subsystem of Eq. (9).

$$\dot{x}(t) = (A(\delta) + BK_{se}(\delta))\, x(t) + Bu_{cl}(t), \tag{9}$$

$$K_{se}(\delta) := [0 \quad K_s(\delta)], \tag{10}$$

where $K_{se}(\delta)$ is defined by using $K_s(\delta)$ because the state variables, x_s and x_{cl}, have different dimension. The fixed feedback controller $u_{cl}(t) = K_{cl}x(t)$ is designed such that the cost function are minimized by $K_s(\delta)$ and K_{cl}. The detailed design method is described from the next section. It is described that the proposed methods of many types are derived as following theorems and corollaries with **Proposition 1** as a base.

Proposition 1 *Consider the following system is given by the following Eq. (11), where $w(t)$ is an impulse disturbance input.*

$$\left\{\begin{array}{l} \dot{x}(t) = (A(\delta) + BK_{se}(\delta))\, x(t) \\ \qquad\qquad + Bu_{cl}(t) + B_w w(t) \\ z(t) = C(\delta)x(t) + Du_{cl}(t) \end{array}\right. \tag{11}$$

$$B_w = I, \quad C(\delta) = \left[\begin{array}{c} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} K_{se}(\delta) \end{array}\right], \quad D = \left[\begin{array}{c} 0 \\ R^{\frac{1}{2}} \end{array}\right]$$

If this H_2 state feedback controller, $u_{cl} = K_{cl}x$ is obtained, the proposed controller of $u = u_{se}(t) + u_{cl}(t) = (K_{se}(\delta) + K_{cl})x = (K_{se}(\delta) + K_{cl})x$ is minimized the the cost function of Eq. (3) for the target plant of (1) .

3.1 First stage design for subsystem

As described in the previous section, two typical examples are considered for designing the GS control system. **Case A.** describes a design method in which the obtained gain is an affine function with respect to the variation parameter. **Case R.** describes a design method in which the obtained gain is a rational function of the variation parameter.

Case 1 A. *The first type of GS controller has gains given by affine functions of parameter variation. The controller is given by following corollary.*

If Q_s is given as a parameter dependent linear function of $Q_s(\delta) = Q_{s0} + \sum_{i=1}^{k} \delta_i Q_{si}$, **Corollary 1 A.** is obtained.

Corollary 1 A. *If there exists $X_s > 0$ and $Y_s(\delta)$ such that Eq. (12) and Eq. (13) are held, the closed-loop system is stabilized by the state feedback $u_s(t) = K_s(\delta)x_s(t) := Y_s(\delta)X_s x_s := (Y_{s1} + \delta Y_{s2})X_s^{-1}x(t)$.*

$$minimize\ \gamma_s,\ subject\ to:$$

$$\left[\begin{array}{ccc} He\left[A_s(\delta)X_s + B_sY_s(\delta)\right] & * & * \\ Q_s(\delta)X_s & -Q_s(\delta) & * \\ R^{\frac{1}{2}}Y_s(\delta) & 0 & -I \end{array}\right] < 0, \tag{12}$$

$$\left[\begin{array}{cc} Z_s & I \\ I & X_s \end{array}\right] > 0,\ trace[Z_s] < \gamma_s, \tag{13}$$

where $Q_s(\delta) > 0$ and $R > 0$ are weighting matrices. Furthermore, though maximising the trace of X_s, J_s is minimized.

If Q_s is given as a parameter dependent quadratic function of $Q_s(\delta) = (Q_{s0} + \sum_{i=1}^{k} \delta_i Q_{si})^2$, the new corollary is obtained by using **Corollary 1 A.** and Schur complement.

Case 1 R. *The second type of GS controller has gains given by polynomial functions of parameter variation. The controller is given by following theorem. Theorem description is omitted for want of space. [1]*

3.2 Second stage design for target system

Here, we explain the second stage after completing the first stage in the two-stage design.

Case 2 A. *When* **Case 1 A.** *is used for the first-stage design, the second-stage design of the proposed method can be described as the following theorem by using the* **Proposition 1**.

Theorem 1 *If there exist $X > 0$ and Y such that Eq. (14) and Eq. (15) held, the closed-loop system is stabilized by the state feedback $u_{cl}(t) = K_{cl}x(t) := YX^{-1}x(t)$.*

$$minimize\ \gamma\ subject\ to:$$

$$\left[\begin{array}{cc} He\left[A(\delta)X + BY\right] & X^TC^T(\delta) + Y^TD^T \\ C(\delta)X + DY & -I \end{array}\right] < 0 \tag{14}$$

$$\left[\begin{array}{cc} Z & I \\ I & X \end{array}\right] > 0,\ trace[Z] < \gamma \tag{15}$$

$$C(\delta) = \left[\begin{array}{c} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} K_{se}(\delta) \end{array}\right], \quad D = \left[\begin{array}{c} 0 \\ R^{\frac{1}{2}} \end{array}\right],$$

where $Q \geq 0$ and $R > 0$ are weighting matrices. Furthermore, though maximizing the trace of X, J is minimized.

Case 2 R. *When* **Case 1 R.** *is used for the first-stage design, $K_s(\delta)$ is given by rational functions. To solve this difficulty, a new argument value, z_δ, is introduced. The equivalent system for the Eq.(11) is defined by using a redundant variables. The second-stage design of the proposed method can be described as a corollary by using Theorem 1 and Linear Fractional Transformation. The description of the corollary is omitted for want to space.*

Remark 2 *$K_s(\delta)$ which is described as rational functions of δ can also be applied to Theorem 1 by using Affine interpolation. The theorem used for the second-stage design is different depending on whether the affine functions give the gains or the rational functions give the ones in the first-stage design. If you chose* **Case 1 A.** *in first-stage design,* **Case 2 A.** *in second-stage design should be used.*

4 The weighting matrices in the first stage design

The gains $K_s(\delta)$ obtained at the first stage in designing the control system is not always such as to minimize the cost function at the second stage. The selection of the weighting matrices in the first stage is the selection of the free parameters in the design of proposed controller. The selection weighting matrices should ensure the performance of the control system at least as high as the Robust Linear Quadratic(R-LQ) controller. The systems in the ILQ method is not need to be considered the same perturbation in the design. A new system is defined for an ILQ method. Let $\hat{\Phi}$ and $\hat{\Phi}_s$ be the total system and the subsystem for the ILQ method, respectively. When they include the perturbation δ explicitly, they are described as $\hat{\Phi}(\delta)$ and $\hat{\Phi}_s(\delta)$. The target system and the subsystem which don't include δ by uniquely determining the value of δ are described as $\hat{\Phi}(N)$ and $\hat{\Phi}_s(N)$. Let $\hat{K}$ and $\hat{K}_s$ be gains for $\hat{\Phi}$ and for $\hat{\Phi}_s$ which are extracted from $\hat{K}$, respectively. When they include the perturbation δ explicitly, they are described as $\hat{K}(\delta)$ and $\hat{K}_s(\delta)$, Let $\hat{K}(N)$ and $\hat{K}_s(N)$ be fixed gains for $\hat{\Phi}(N)$ and $\hat{\Phi}_s(N)$. Now, we propose Pseudo ILQ. ILQ method [4] is naturally extend to the framework of suboptimal problem as following Corollary 2.

Corollary 2 F. *If there exist $\hat{P}_s = \hat{P}_s > 0$ and $\epsilon > 0$ such that the following equation is hold, $Q_s(N) = -\hat{P}_s\hat{A}_s(N) - \hat{A}_s(N)\hat{P}_s + \hat{K}_s(N)\hat{K}_s(N)$ is obtained. We call this corollary Pseudo ILQ.*

$$\begin{gathered} minimize\ \epsilon,\ subject\ to: \\ \begin{bmatrix} S(N) & \hat{P}_s\hat{B}_s + \hat{K}_s(N) \\ \hat{B}_s\hat{P}_s + \hat{K}_s(N) & -\epsilon \end{bmatrix} < 0 \qquad (16) \\ S(N) = \hat{P}_s\hat{A}_s(N) + \hat{A}_s(N)\hat{P}_s - \hat{K}_s(N)\hat{K}_s(N) \end{gathered}$$

where the $\hat{K}_s(N)$ is a stabilizing feedback gain and $(Q_s^{\frac{1}{2}}(N), \hat{A}_s(N))$ is detectable.

Although it is ideal if ϵ becomes 0, that can not always be guaranteed due to numerical adverse conditions such as parameter variation δ. However, what we should obtain is not a strict ILQ solution. Our aim is to obtain the first stage Q_s of the cost function which is convenient in the second stage. It is to propose an ILQ-like corollary that gives priority to answering. However, the value of ϵ is important. The smaller the ϵ value is, the closer it is to the strict ILQ solution. It can be said that there is a high possibility that Q_s of the first stage which is convenient in the second stage is obtained. As the ϵ value increases, it is more likely that the Q_s is away from the ILQ answer and the required Q_s is not obtained. The following the procedure gives an outline of the selection procedure of the weighting matrix for the proposed method using the above Corollary.

Procedure 1 *Following outline of the selection procedure of Q_s for the proposed method are described in the order of 1 to 3.*

1. *Design the LQ control system with stabilizing gains for the target plant $\hat{\Phi}$.*
2. *Extract the part of the closed-loop subsystem from the closed-loop target plant. Note that the obtained closed-loop system of the subsystem is stable.*
3. *The feedback gains $\hat{K}_s$ that stabilizes this subsystem $\hat{\Phi}_s$ and the subsystem are applied to ILQ method or corollaries.*

Considering the robustness of the controlled object in each step of the above procedure, seven kinds of procedures are obtained.

5 Numerical example

The proposed controller is applied to the Jib crane system.

5.1 Mathematical model of Jib Crane

In this section, the dynamics of a crane is derived as Eq. (17)-(18) from Euler-Lagrange equations. A schematic diagram of the crane is shown in Figure 1. The physical constants and variables are shown in Table 1.

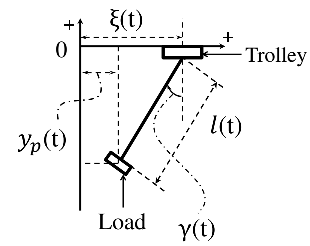

Figure 1: Schematic diagram for crane system

Table 1: Physical parameters

$m_p\ [m]$	Mass of payload
$m_t\ [kg]$	Mass of trolley
$l\ [m]$	lope length
$g\ [m/s^2]$	Acceleration of gravity
$K_{g,j}$	Jib motor gear ratio
$K_{t,j}$	Jib motor torque constant

$$\begin{cases} \ddot{\xi} = \frac{m_p}{m_j - m_p}\left(\ddot{l} - g\right)\gamma + \frac{K_{g,j}}{m_j - m_p}K_{t,j}I_j \\ y_p = \xi - l\sin\gamma \end{cases} \qquad (17)$$

$$(m_j - m_p)l\ddot{\gamma} = (m_p\ddot{l} - m_j g)\gamma - 2\dot{l}\dot{\gamma}(m_j - m_p)l + K_{t,j}I_j \qquad (18)$$

5.2 Partial GS controller synthesis

The state variable $x(t) = [w_p\,\xi\,\dot{\xi}\,\gamma\,\dot{\gamma}]$ and $x_s(t) = [\gamma\,\dot{\gamma}]$ are defined. The descriptor representation can be described as Eq.(19) and Eq.(20).

$$E\dot{x}(t) = A(\delta)x(t) + Bu(t) \qquad (19)$$

$$E_s(\delta)\dot{x}_s(t) = A_s(\delta)x_s(t) + B_s u_s(t) \qquad (20)$$

If the coefficient matrix E is a regular matrix, section 3 - 4 can be expanded to the descriptor system, easily from some previous studies [2] [3]. $E_s(\delta)$, $A_s(\delta)$, B_s are controllable. The proposed controller can be designed for Eq.(19). Hence, let $x_{ds}(t) = [\gamma\,\dot{\gamma}\,\dot{\gamma}\,\ddot{\gamma}]$ as descriptor variables. Most of the dynamical system can be designed with the argument of state variable, $x_d = [x\,\dot{x}]$. However, It is reported that the argument of state variable can be reduced as $x_{ds}(t) = [\gamma\,\dot{\gamma}\,\dot{\gamma}\,\ddot{\gamma}] \Rightarrow [\gamma\,\dot{\gamma}\,\ddot{\gamma}]$ [3]. The argument system Eq.(19) and Eq.(20) is obtained. The lower bounds and upper bounds of l, $\dot{l}$ and $\ddot{l}$ are assumed as :

$$\delta_1 = l \in \left[\underline{l},\ \bar{l}\right],\ \delta_2 = \dot{l} \in \left[\underline{\dot{l}},\ \bar{\dot{l}}\right],\ \delta_3 = \ddot{l} \in \left[\underline{\ddot{l}},\ \bar{\ddot{l}}\right].$$

δ_1 and δ_3 are treated as uncertain parameters. δ_2 is treated as a scheduling parameter. To derive the stabilizing state feedback controller, the cost function (4) is considered. Let l, $\dot{l}$ and $\ddot{l}$ are the lope length, the expansion and contraction velocity and that of acceleration. The proposed controller is obtained by solving LMI conditions shown in **Theorem 2, Corollary 1 A.** and **Corollary 2 F.** at each vertex matrices

5.3 Simulation result

In this study, the effectiveness of the proposed controller is compared with robust LQ controller (R-LQ) and GS controller via single Lyapunov function (SL-GS) in the simulations. The R-LQ guarantees the robustness for the variation of the velocity of the lope length perturbation by fixed controller gain. The SL-GS guarantees the robustness for the variation of the velocity of the lope length perturbation by varying controller gain. The physical constants of crane system are given as Table 2. In this study, the lope length is started from

Table 2: The numerical value of physical parameters

$m_p\ [kg]$	0.99	$K_{t,j}$	15.5	$g\ [m/s^2]$	9.81
$m_t\ [kg]$	0.60	$K_{g,j}$	3.70		

0.1 [m] to 0.7 [m] with the velocity of 0.260 [m/s] and the acceleration of 4 [m/s^2] at 1 [s]. The trolley position moves from 0 [m] to 0.3 [m] with the velocity of 0.2 [m/s]. The upper bounds and lower bounds are defined as follows.

$$l \in [0.10,\ 0.70]\,,\ \dot{l} \in [-0.260,\ 0.2650]\,,\ \ddot{l} \in [-5.0,\ 5.0]$$

The upper bound values of the cost function were obtained as shown in the following table. The Proposed

Table 3: The upper bound value of the cost functions

	R-LQ	SL-GS	Proposed
upper bound value	39.0773	24.8253	26.1462

controller upper bounds of the cost function are nearly identical to SL-GS, and it can be seen from this that the same performance as SL-GS can be guaranteed. The left graph of Figure 2 shows the horizontal load position of crane. The proposed method has the same performance as the conventional SL-GS, which shows that it is much better performance than R-LQ, by the upper bound of the cost function value and simulation. The cumulative

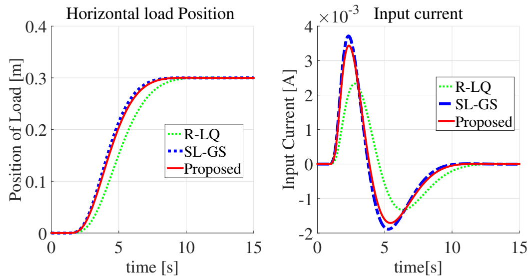

Figure 2: The load position (left) and the input current (right).

calculation time between input and output with each controller is obtained. This cumulative calculation time is obtained 1000 times for each controller. That of the average is obtained as Table 4. The proposed method has been able to reduce the computational complexity compared with SL-GS.

Table 4: Amount of the cumulative calculation time

Controller	RLQ	SL-GS	Proposed
times[s] $\times 10^{-2}$	3.12	5.17	4.13

6 Conclusion

This paper proposed the new concept of the GS controller and the design methods. Several theorems, corollaries, methods, and procedures for that is obtained. This controller can be applied to many systems which has the certain structure. We made the structure clear. Furthermore, by extending the theory of ILQ, an effective method of determining free parameters included in the proposed method was proposed. The proposed controller has varying gains for only the state variable of the subsystem. By the simulation, It is verified that the proposed controller has almost the same performance as the ordinary GS, and numerical results showing better performance than robust LQ were obtained.

References

[1] A. Packard, G. Becker, "Quadratic Stabilization of Parametrically-Dependent Linear Systems using Parametrically-Dependent Linear, Dynamic Feedback.", Advances in Robust and Nonlinear Control Systems DSC-Vol. 43, pp, 29-36, 1992

[2] A. Sanbayashi, M. Narita, G. Chen, I. Takami, "Gain scheduled control for active magnetic bearing system considering gyroscopic effect, Information Technology and Electrical Engineering (ICITEE), 2015 7th International Conference on, pp. 297-303, 2015

[3] T. Kanada, Y. Watanabe, G. Chen, "Robust $\mathcal{H}_2$ Control for Two-Wheeled Inverted Pendulum Using LEGO Mindstorms", Australian Control Conference, Engineers Australia, ThIT3.2, pp.136-141, 2011

[4] T. Fujii, "The Inverse Problem of Optimal Regulator", Journal of the Society of Instrument and Control Engineers, Vol. 27, No. 8 , pp.717-726, 1988 (in Japanese)